Exercise 2.5.6

(The leaky bucket) The following example (Hubbard and West 1991, p. 159) shows that in some physical situations, non-uniqueness is natural and obvious, not pathological.

Consider a water bucket with a hole in the bottom. If you see an empty bucket with a puddle beneath it, can you figure out when the bucket was full? No, of course not! It could have finished emptying a minute ago, ten minutes ago, or whatever. The solution to the corresponding differential equation must be non-unique when integrated backwards in time.

Here's a crude model of the situation. Let $h(t)$ = height of the water remaining in the bucket at time t; a = area of the hole; A = cross-sectional area of the bucket (assumed constant); $v(t)$ = velocity of the water passing through the hole.

a) Show that $av(t) = A\dot{h}(t)$. What physical law are you invoking?

b) To derive an additional equation, use conservation of energy. First, find the change in potential energy in the system, assuming that the height of the water in the bucket decreases by an amount Δh and that the water has density ρ. Then find the kinetic energy transported out of the bucket by the escaping water. Finally, assuming all the potential energy is converted into kinetic energy, derive the equation $v^2 = 2gh$.

c) Combining (a) and (b), show that $\dot{h} = -C\sqrt{h}$, where $C = \sqrt{2g}\left(\frac{a}{A}\right)$.

d) Given $h(0) = 0$ (bucket empty at $t = 0$), show that the solution for $h(t)$ is non-unique *in backwards time*, i.e., for $t < 0$.

[Actually, yes you can because of evaporation. You can measure the volume of water remaining on the floor, refill the bucket, let all the water leak again, and use a stopwatch to find how long it takes for the same volume to remain on the floor. Assume there's no evaporation in this exercise.]

Solution

By conservation of mass, the rate of mass flowing through a cross-section of the bucket must be equal to the rate of mass flowing out through the hole.

$$\left.\frac{dm}{dt}\right|_{\text{Mass flow rate through } A} = \left.\frac{dm}{dt}\right|_{\text{Mass flow rate through } a}$$

The mass flow rate is equal to the density times the volumetric flow rate. The density of water ρ is the same in A as it is in a.

$$\rho\left.\frac{dV}{dt}\right|_{\text{Volumetric flow rate through } A} = \rho\left.\frac{dV}{dt}\right|_{\text{Volumetric flow rate through } a}$$

The volume flow rate is the fluid velocity times the area the fluid is flowing through. If $h(t)$ is the position of the fluid level in the bucket, then $h' = \dot{h}$ is the velocity that the fluid level is moving.

$$\rho[Ah'(t)] = \rho[av(t)]$$

Therefore, dividing both sides by ρ,

$$Ah'(t) = av(t). \tag{1}$$

Assuming that the water is incompressible (constant density), steady (laminar flow), and inviscid (no internal friction), Bernoulli's equation applies.

$$p_1 + \rho g y_1 + \frac{1}{2}\rho v_1^2 = p_2 + \rho g y_2 + \frac{1}{2}\rho v_2^2$$

Let point 1 be at the height of the water in the bucket, and let point 2 be in the hole at the bottom of the bucket. As both points are exposed to the air, the pressures are equal to that of the atmosphere.

$$\begin{aligned} p_{\text{atm}} + \rho g[h(t)] + \frac{1}{2}\rho[h'(t)]^2 &= p_{\text{atm}} + \rho g(0) + \frac{1}{2}\rho[v(t)]^2 \\ \rho g h(t) &= \frac{1}{2}\rho\{[v(t)]^2 - [h'(t)]^2\} \\ &= \frac{1}{2}\rho\left\{[v(t)]^2 - \left[\frac{av(t)}{A}\right]^2\right\} \\ &= \frac{1}{2}\rho\left[1 - \left(\frac{a}{A}\right)^2\right][v(t)]^2 \end{aligned}$$

Since the area of the hole is much smaller than the cross-sectional area of the bucket, $a \ll A$.

$$\rho g h(t) = \frac{1}{2}\rho[v(t)]^2$$

Therefore, multiplying both sides by $2/\rho$,

$$[v(t)]^2 = 2gh(t).$$

Take the square root of both sides to solve for $v(t)$.

$$v(t) = \pm\sqrt{2gh(t)}$$

Since the water exits the bottom, the velocity is downward in the negative direction.

$$v(t) = -\sqrt{2gh(t)}$$

Use equation (1).

$$-\sqrt{2gh(t)} = \frac{Ah'(t)}{a}$$

Therefore, multiplying both sides by a/A,

$$h'(t) = -\frac{a}{A}\sqrt{2g}\sqrt{h(t)}.$$

A solution to this ODE for $h(t)$ can be obtained by separating variables.

$$\frac{dh}{dt} = -\frac{a}{A}\sqrt{2g}\,h^{1/2}, \quad h(0) = 0 \tag{2}$$

$$h^{-1/2}\,dh = -\frac{a}{A}\sqrt{2g}\,dt$$

Integrate both sides.

$$\int h^{-1/2}\,dh = -\int \frac{a}{A}\sqrt{2g}\,dt$$

$$2h^{1/2} = -\frac{a}{A}\sqrt{2g}\,t + C \tag{3}$$

Apply the initial condition $h(0) = 0$ now to determine C.

$$2(0)^{1/2} = -\frac{a}{A}\sqrt{2g}\,(0) + C \quad \rightarrow \quad C = 0$$

As a result, equation (3) becomes

$$2h^{1/2} = -\frac{a}{A}\sqrt{2g}\,t.$$

Divide both sides by 2.

$$h^{1/2} = -\frac{a}{A}\sqrt{\frac{g}{2}}\,t$$

Notice that the left side is nonnegative, so the domain is $t \leq 0$. Square both sides to solve for $h(t)$.

$$h(t) = \frac{a^2 g}{2A^2}t^2, \quad t \leq 0$$

Notice that $h(t) = 0$ also satisfies the ODE and initial condition in equation (2), so the solution is not unique for $t < 0$.